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the hyperbola, so that t' and f' coincide, and cf the intercept of the produced normal at P to the hyperbola, or tangent to the ellipse; so that t and f coincide.

Then, since the proportion $Ct'.Cn=BC^2$ [Cn being the abscissa, and Ct' the intercept of the tangent on the minor axis] belongs to the general conic; and since $Ct'=Ct=Cf$; and $Cn=Cz$ for both the confocals; therefore the third magnitude CB^2 must also be fixed and equal in both confocal curves; and $2CB$ be the common minor axis of ellipse and hyperbola.

NOTE ON THE EXPANSION OF DEVERTEBRATE DETERMINANTS.

By ORLANDO S. STETSON, Syracuse University.

In a paper entitled *"Co-axial Minors of a Determinant of the Fourth Order," Dr. Muir obtains what might be regarded as a sort of converse of Cayley's Expansion Theorem, viz., a general law for the expansion of an invertebrate co-axial minor in terms of the vertebrate co-axial minors of a given determinant and the elements of the principal diagonal. Dr. Muir mentions here, also, the possibility of presenting this as the ultimate case of a more general law, namely, a law for the expansion of a devertebrated† determinant in terms of the elements of the principal diagonal of the given determinant and their co-axial minors.

My attention has been drawn by Dr. William H. Metzler to the desirability of obtaining, if possible, an explicit statement of such a general law.

The object of this paper is to show how such a general law may be deduced. A slight modification of method will present, algebraically, a general proof of Cayley's Expansion Theorem given in Muir's Theory of Determinants.‡

Let $D=|a_{ij}|$ represent a general determinant of order n and let $\triangle$ be a determinant of the following form:

$$\triangle=\begin{vmatrix} a_{11}+x_{11} & a_{12}\cdots\cdots a_{1k}\cdots\cdots a_{1n} \\ a_{21} & a_{22}+x_{22}\cdots a_{2k}\cdots\cdots a_{2n} \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ a_{k1} & a_{k2}\cdots\cdots a_{kk}+x_{kk}\cdots a_{kn} \\ a_{n1} & a_{n2}\cdots\cdots a_{nk}\cdots\cdots a_{nn}+x_{nn} \end{vmatrix}.$$

Expanding $\triangle$ as a determinant with binomial elements, the term independent of the variables $x_{11}, x_{22}, \cdots\cdots, x_{nn}$ will be the given determinant D.

The coefficients of the variables $x_{11}, x_{22}, \cdots\cdots, x_{nn}$, are, respectively, the principal minors of order $(n-1)$ which we shall denote by $A_{\substack{1\\1}}, A_{\substack{2\\2}}, \cdots\cdots, A_{\substack{n\\n}}$. Similarly, the coefficients of $x_{11}x_{22}, x_{11}x_{33}, \cdots\cdots, x_{n-1}x_n$, are, respectively, the principal minors of order $(n-2)$ which we designate by $A_{\substack{12\\12}}, A_{\substack{13\\13}}, \cdots\cdots, A_{\substack{n-1,n\\n-1,n}}$; etc.

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*Transactions of the Royal Society of Edinburgh, Vol. 39, No. 10.

†k elements of principal diagonal zero, $n-k$ not zero.

‡Muir, p. 85.

The expansion may therefore be written as follows:

$$\triangle = D + \Sigma x_{11} A_{\substack{1\\1}} + \Sigma x_{11} x_{22} A_{\substack{12\\12}} + \Sigma x_{11} x_{22} x_{33} A_{\substack{123\\123}}$$
$$+ \cdots\cdots + \Sigma x_{11} x_{22} \cdots\cdots x_{kk} A_{\substack{12\ldots k\\12\ldots k}} + \cdots\cdots + x_{11} x_{22} \cdots\cdots x_{nn}.$$

Now, let $\triangle_k$ be a devertebrated determinant of which the k zero elements in any order are $a_{r_1 r_1}, a_{r_2 r_2}, \cdots, a_{r_k r_k}$. Substituting in $\triangle$ for $x_{r_1 r_1}, x_{r_2 r_2}, \cdots, x_{r_k r_k}$, the values $-a_{r_1 r_1}, -a_{r_2 r_2}, \cdots, -a_{r_k r_k}$, and placing equal to zero the remaining $n-k$ variables, it becomes evident that the expansion of the devertebrate will take the following form:

$$\triangle_k = D - \Sigma a_{r_1 r_1} A_{\substack{r_1\\r_1}} + \Sigma a_{r_1 r_1} a_{r_2 r_2} A_{\substack{r_1 r_2\\r_1 r_2}} + \cdots + (-1)^{(k)\lambda}\, \Sigma a_{r_1 r_1} a_{r_2 r_2} \cdots a_{r_\lambda\, r_\lambda} A_{\substack{r_1 r_2 \ldots r_\lambda\\r_1 r_2 \ldots r_\lambda}}$$
$$+ \cdots + (-1)^{(k)k} a_{r_1 r_1} a_{r_2 r_2} \cdots a_{r_k r_k} A_{\substack{r_1 r_2 \ldots r_k\\r_1 r_2 \ldots r_k}}.$$

We obtain, consequently, the following theorem:

A devertebrated determinant of order n, containing k zero elements may be developed with a series of terms, the first of which is the given determinant of order n, the next k terms being the products of each of the k elements of the given determinant into its co-axial minor, the next $\frac{1}{2}k(k-1)$ *terms being the products of each pair of the k elements of the given determinant into its co-axial minor, etc. The signs of the respective terms are negative or positive according as the combination of the k elements are odd or even.*

It is worthy of notice that for $k=n$, the expansion contains the law for the expression of an *invertebrate* co-axial minor in terms of the vertebrate co-axial minors of the given determinant and its principal diagonal elements.

For this case the expansion formula may be expressed more conveniently as follows:

$$\triangle_n = D + \Sigma x_{11} \triangle_{\substack{1\\1}} + \Sigma x_{11} x_{22} \triangle_{\substack{12\\12}} + \Sigma x_{11} x_{22} x_{33} \triangle_{\substack{123\\123}}$$
$$+ \cdots\cdots + \Sigma x_{11} x_{22} \cdots\cdots x_{n-2} A_{\substack{12\ldots(n-2)\\12\ldots(n-2)}} + (-1)^{n-1}(n-1) x_{11} x_{22} \cdots\cdots x_{nn}.$$

CAYLEY'S EXPANSION THEOREM.

Let D represent a given determinant of order n, and let an identity be formed as follows:

$$D = \begin{vmatrix} (a_{11}-x_{11})+x_{11} & a_{12} & a_{1n} \\ a_{21} & (a_{22}-x_{22})+x_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & (a_{nn}-x_{nn})+x_{nn} \end{vmatrix}.$$

Expanding as a determinant with binomial elements and letting $\triangle$ represent the determinant formed from the general determinant by subtracting from the elements of the principal diagonal, $x_{11}, x_{22}, \cdots\cdots, x_{nn}$, we have:

$$D=\Delta+\Sigma x_{11}\Delta_{\substack{1\\1}}+\Sigma x_{11}x_{22}\Delta_{\substack{12\\12}}+\Sigma x_{11}x_{22}x_{33}\Delta_{\substack{123\\123}}+\cdots\cdots$$
$$+\cdots\cdots+\Sigma x_{11}x_{22}\cdots\cdots x_{kk}\Delta_{\substack{12\ldots k\\12\ldots k}}+\cdots\cdots+x_{11}x_{22}\cdots\cdots x_{nn}.$$

Now, placing $x_{11}=a_{11}$, $x_{22}=a_{22}$, $\cdots\cdots$, $x_{nn}=a_{nn}$, Cayley's expansion is an immediate consequence since the principal diagonals of all the Δ's become zero.

As illustrations we may assume—

$$\Delta=\begin{vmatrix} a_{11}+x_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}+x_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}+x_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}+x_{44} \end{vmatrix}.$$

Expanding

$$\Delta=D+x_{11}A_{\substack{1\\1}}+x_{22}A_{\substack{2\\2}}+x_{33}A_{\substack{3\\3}}+x_{44}A_{\substack{4\\4}}$$
$$+x_{11}x_{22}A_{\substack{12\\12}}+x_{11}x_{33}A_{\substack{13\\13}}+x_{11}x_{44}A_{\substack{14\\14}}$$
$$+x_{22}x_{33}A_{\substack{23\\23}}+x_{22}x_{44}A_{\substack{24\\24}}+x_{33}x_{44}A_{\substack{34\\34}}$$
$$+x_{11}x_{22}x_{33}A_{\substack{123\\123}}+x_{11}x_{22}x_{44}A_{\substack{124\\124}}$$
$$+x_{11}x_{33}x_{44}A_{\substack{134\\134}}+x_{22}x_{33}x_{44}A_{\substack{234\\234}}+x_{11}x_{22}x_{33}x_{44}.$$

For $k=1$, assume $x_{22}=-a_{22}$, and place $x_{11}=x_{33}=x_{44}=0$, then,

$$\Delta_1=D-a_{22}A_{\substack{2\\2}}.$$

For $k=2$, assume $x_{11}=-a_{11}$, $x_{44}=-a_{44}$, and place $x_{22}=x_{33}=0$, then

$$\Delta_2=D-a_{11}A_{\substack{1\\1}}-a_{44}A_{\substack{4\\4}}+a_{11}a_{44}A_{\substack{14\\14}}.$$

For $k=3$, assume $x_{11}=-a_{11}$, $x_{22}=-a_{22}$, $x_{33}=-a_{33}$ and place $x_{44}=0$, then

$$\Delta_3=D-a_{11}A_{\substack{1\\1}}-a_{22}A_{\substack{2\\2}}-a_{33}A_{\substack{3\\3}}+a_{11}a_{22}A_{\substack{12\\12}}+a_{11}a_{33}A_{\substack{13\\13}}$$
$$+a_{22}a_{33}A_{\substack{23\\23}}-a_{11}a_{22}a_{33}.$$

For $k=4$, assume $x_{11}=-a_{11}$, $x_{22}=-a_{22}$, $x_{33}=-a_{33}$, $x_{44}=-a_{44}$, then

$$\Delta_4=D-a_{11}A_{\substack{1\\1}}-a_{22}A_{\substack{2\\2}}-a_{33}A_{\substack{3\\3}}-a_{44}A_{\substack{4\\4}}+a_{11}a_{22}A_{\substack{12\\12}}+a_{11}a_{33}A_{\substack{13\\13}}$$
$$+a_{11}a_{44}A_{\substack{14\\14}}+a_{22}a_{33}A_{\substack{23\\23}}+a_{22}a_{44}A_{\substack{24\\24}}+a_{32}a_{44}A_{\substack{34\\34}}-3a_{11}a_{22}a_{33}a_{44}.$$

SYRACUSE UNIVERSITY, *March 20, 1904.*